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ANALYSIS OF SEMI-MARKOV SYSTEMS WITH FUZZY INITIAL DATA

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Abstract

In real operating conditions of complex systems, random changes in their possible states occur in the course of their operation. The traditional approach to describing such systems uses Markov models. However, the real non-deterministic mechanism that controls the duration of the system's stay in each of its possible states predetermines the insufficient adequacy of the models obtained in this case. This circumstance makes it expedient to consider models that are more general than Markov ones. In addition, when choosing such models, one should take into account the fundamental often manifested feature of the statistical material actually used in the processing of an array of observations, their small sample. All this, taken together, makes it relevant to study the possibility of developing less demanding, tolerant models of the behavior of complex systems. A method for the analysis of systems described under conditions of initial data uncertainty by semi-Markov models is proposed. The main approaches to the description of this uncertainty are considered: probabilistic, fuzzy, and bi-fuzzy. A procedure has been developed for determining the membership functions of fuzzy numbers based on the results of real data processing. Next, the following tasks are solved sequentially. First, the vector of stationary state probabilities of the Markov chain embedded in the semi-Markov process is found. Then, a set of expected values for the duration of the system's stay in each state before leaving it is determined, after which the required probability distribution of the system states is calculated.

The proposed method has been developed to solve the problem in the case when the parameters of the membership functions of fuzzy initial data cannot be clearly estimated under conditions of a small sample.

Keywords: semi-Markov models of systems, bi-fuzzy input data, calculation of state probability distributions.

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1. Introduction

Let's consider the problem of analyzing a system whose operation is described by a semi-Markov process (SMP). Let this SMP be given by the matrix $F(t)$ of conditional distribution densities of the random duration of the system's stay in state i before transition to state j.

$$F(t)=\{f_{ij}(t)\},\ i=1,2,...,n,\ j=1,2,...,n,$$

as well as the probability distribution of the system being in possible states, which is determined by the transition matrix of the Markov chain embedded in the SMP – $P = \{p_{ij}\}$.

The standard problem of calculating the final probability distribution of SMP states is divided into two sequentially solved subtasks [1].

Subproblem 1. Calculation of the probability distribution of states for an embedded Markov chain (EMC).

Let's introduce the desired vector $\pi = (\pi_1, \pi_2,..., \pi_n)$, which specifies the final probability distribution of the states of the EMC, which is determined by solving the system of linear algebraic equations (SLAE):

$$\begin{cases} \pi = \pi P, \\ \sum_{j=1}^{n} \pi_j = 1. \end{cases} \quad (1)$$

Let the solution of this SLAE be the vector:

$$\pi = \left(\pi_1^{(0)}, \pi_2^{(0)}, ..., \pi_n^{(0)}\right).$$

Subproblem 2. Calculation of the average duration of the process in each of the possible states before leaving for any other state:

$$\bar{T}_{ij} = \int_0^\infty t f_{ij}(t)\mathrm{d}t, \; i = 1,2,...,n, \; j = 1,2,...,n,$$

$$\bar{T}_i = \sum_{j=1}^{n} p_{ij}\bar{T}_{ij}, \; i = 1,2,...,n, \; j = 1,2,...,n. \quad (2)$$

Let's now introduce the vector $v = (v_1, v_2,..., v_n)$, which determines the required probability distribution of the system states. The components of this vector, in accordance with [2–5], are calculated by the formula:

$$v_j = \frac{\pi_j T_j}{\sum_{j=1}^{n} \pi_j T_j}, \; i = 1,2,...,n. \quad (3)$$

The technology for analyzing a system whose behavior is described by the SMP becomes much more complicated under conditions of uncertainty, for example, if the model parameters are not clearly specified [6–8].

2. Materials and methods

Let's briefly analyze the known results. In [9], restrictions on existing approaches to the construction of semi-Markov fuzzy models of systems are considered. A solution is proposed for the problem of finding a stationary probability distribution for a system whose behavior is not clearly defined by a matrix of conditional distribution functions for the duration of the system's stay in states and a matrix of transition probabilities for an embedded Markov chain. However, it is assumed that the fuzzy initial data are given at intervals. The level of adequacy of the resulting solution cannot be assessed, since this level cannot be correctly assessed to describe the initial data and intermediate results. In particular, it is not clear how to interpret the obtained vector of the stationary distribution of states of an embedded Markov chain, in the description of which the set of transition probabilities does not satisfy the normalization conditions.

The same interval approach, which uses the formal replacement of the probabilities of the system states and the time the system spends in the corresponding states by interval fuzzy numbers, is proposed in [10]. The error of the results obtained in this case is not discussed. In [11], the fuzzy values of the system transition probabilities are also displayed in intervals. And here the error of the results is not discussed. In [12], the problem is solved approximately using the Markov approximation of the real process. In [13, 14], the simplest interval model is again adopted to describe the membership function of fuzzy initial data. This approach is motivated, since it allows to reduce the original problem with uncertainty to two boundary particular problems. In the first task, all fuzzy parameters are given by the values of the left boundaries of their fuzziness intervals, and in the second problem, by the values of the right boundaries of these intervals. It is clear that such a solution to the original problem with an unpredictable inaccuracy of the result obtained in many situations does not meet the needs of practice. In accordance with this, the goal of the study is formulated – the development of a method for the analysis of semi-Markov systems for fuzzy given initial data. To achieve this goal, it is necessary to solve the following tasks:

1. Development of a methodology for determining membership functions of fuzzy system parameters based on sets of their observed values.

2. Development of a method for calculating the probability distribution of the states of a semi-Markov system, the parameters of which are determined fuzzy.

Let's turn to the consideration of methods for solving the formulated problems.

3. Results and discussion

Let a set of values $x=(x_1, x_2, ..., x_n)$ of some fuzzy number x be given (or experimentally obtained). Let's find the membership function of this number. For definiteness, let's look for this function in the class of triangular numbers of (L–R)-type. In accordance with this, let's introduce:

$$\mu(x)=\begin{cases} 0,\ x<a, \\ \dfrac{x-a}{b-a},\ a\le x<b, \\ \dfrac{c-x}{c-b},\ b\le x\le c, \\ 0,\ x>c. \end{cases} \tag{4}$$

The computational procedure for estimating the parameters of the membership function (4) is divided into three stages.

Stage 1. Calculation of the modal value of the fuzzy number x. The desired value is defined as the average value for numbers from the set x, that is:

$$b=\frac{1}{n}\sum_{i=1}^{n}x_i. \tag{5}$$

Stage 2. Estimation of the parameters of the left branch of the membership function (4). Since one of the two parameters defining the left branch of function (4) is defined by relation (5), it remains to calculate only the parameter a. Let's solve this problem by the maximum likelihood method.

The equation of the straight line passing through the points (a, 0) and (b, 1) and defining the left branch of the required membership function has the form:

$$f(x)=\frac{(x-a)}{(b-a)}. \tag{6}$$

Then the likelihood function has the form:

$$P_{\wedge}(x)=\prod_{i=1}^{n_{\wedge}}\left(\frac{x_i-a}{b-a}\right),\ n_{\wedge}=\arg\max_{k}\{x_k\le b\}. \tag{7}$$

In this case, the logarithmic likelihood function is determined by the relation:

$$L_{\wedge}(x)=\lg P_{\wedge}(x)=\sum_{i=1}^{n_{\wedge}}\lg(x_i-a)-n_{\wedge}\lg(b-a). \tag{8}$$

Let's determine the value of a that maximizes (8).

$$\frac{dL_{\wedge}(x)}{da}=-\sum_{i=1}^{n_{\wedge}}\frac{1}{x_i-a}+\frac{n_{\wedge}}{b-a}=0. \tag{9}$$

From here let's obtain the equation for calculating a:

$$b-a=\frac{n_{\wedge}}{\sum_{i=1}^{n_{\wedge}}\frac{1}{x_i-a}}. \tag{10}$$

The obvious difficulties of the analytical solution of the nonlinear equation (10) with respect to a lead to the need to use numerical methods. The construction of equation (10) makes it possible to apply the method of simple iteration to its solution [15]. Equation (10) is transformed into the form:

$$a = b - \frac{n_{\wedge}}{\sum_{i=1}^{n_{\wedge}} \frac{1}{x_i - a}} = T_{\wedge}(a). \tag{11}$$

The solution is carried out as follows. In the interval of admissible values $a \in (0; b)$, an arbitrary initial a_1 is chosen. Now let's organize an iterative process of the form:

$$a_{k+1} = T_{\wedge}(a_k), \; k = 1, 2, \ldots \tag{12}$$

where a_k – the value of a obtained after the k-th iteration.

The iterative procedure converges to a stop when the inequality:

$$|a_{s+1} - a_s| < \varepsilon,$$

ε – given small value.

Stage 3. At this stage, the unknown parameter c of the right branch of the membership function (4) is estimated. In this case, the likelihood function is formed:

$$P_{\wedge} = \prod_{i=n_{\wedge}+1}^{n} \left(\frac{c - x_i}{c - b} \right). \tag{13}$$

The log-likelihood function has the form:

$$L_n(x) = \lg P_n(x) = \sum_{i=n_{\wedge}+1}^{n} \lg(c - x_i) - (n - n_{\wedge}) \lg(c - b). \tag{14}$$

Similarly to the previous one, the value c that maximizes (14) is found by solving the equation:

$$\sum_{i=n_{\wedge}+1}^{n} \frac{1}{c - x_i} - \frac{n - n_{\wedge}}{c - b} = 0. \tag{15}$$

Just as before, equation (15), which is nonlinear with respect to c, is solved by a simple iteration method using the procedure:

$$c_{k+1} = T_{\Pi}(c_k), \tag{16}$$

$$T_{\Pi} = b + \frac{n - n_{\wedge}}{\sum_{i=n_{\wedge}+1}^{n_{\wedge}} \frac{1}{c - x_i}}. \tag{17}$$

Procedure (17) converges to a stop when the inequality:

$$\left| c_{s+1} - c_s \right| < \varepsilon.$$

The obtained estimates <a, b, c> of the parameters of the membership function of the number x make it possible to calculate an estimate of its expected value. For this purpose, for the membership function (4) of a fuzzy number x, let's introduce its probabilistic analogue – the conditional distribution density of the number x in accordance with the relation:

$$f(x) = \frac{\mu(x)}{\int_0^{\infty} \mu(x) \mathrm{d}x}, \quad x \geq 0. \tag{18}$$

It is clear that the function (18) has all the necessary properties of the distribution density: this function is non-negative on the entire real axis and the integral of it over its domain of definition is equal to 1. Let's find this function. First define:

$$\int_0^\infty \mu(x)\mathrm{d}x = \int_a^b \frac{x-a}{b-a}\mathrm{d}x + \int_b^c \frac{c-x}{c-b}\mathrm{d}x = \frac{1}{b-a}\int_0^{b-a} u\mathrm{d}u + \frac{1}{c-b}\int_0^{c-b} v\mathrm{d}v = \frac{b-a}{2} + \frac{c-b}{2} = \frac{c-a}{2}. \quad (19)$$

The resulting relation (19) corresponds to the standard formula for calculating the area of a triangle with a base (*c-a*) and a height equal to one. Then:

$$f(x) = \begin{cases} \dfrac{2(x-a)}{(b-a)(c-a)}, & a \le x < b, \\ \dfrac{2(c-x)}{(c-b)(c-a)}, & b \le x \le c. \end{cases} \quad (20)$$

Wherein:

$$\int_a^c f(x)\mathrm{d}x = \frac{2}{(b-a)(c-a)}\int_a^b (x-a)\mathrm{d}a + \frac{2}{(c-b)(c-a)} = \int_b^c (c-x)\mathrm{d}x = \frac{b-a}{c-a} + \frac{c-b}{c-a} = 1, \quad (21)$$

which is what was required.

Now, in accordance with the rules of probability theory, let's determine the expected value for x. There is:

$$\begin{aligned}
\bar{x} &= \int_a^c x f(x)\mathrm{d}x = \frac{2}{(b-a)(c-a)}\int_a^b x(x-a)\mathrm{d}x + \frac{2}{(c-b)(c-a)}\int_b^c x(c-x)\mathrm{d}x = \\
&= \frac{2}{(b-a)(c-a)}\left[\int_a^b x^2\mathrm{d}x + a\int_a^b x\mathrm{d}x\right] + \frac{2}{(c-b)(c-a)}\left[-\int_b^c x^2\mathrm{d}x + a\int_b^c x\mathrm{d}x\right] = \\
&= \frac{\left(b^3-a^3\right)2}{3(b-a)(c-a)} - \frac{2a\left(b^2-a^2\right)}{2(b-a)(c-a)} - \frac{\left(c^3-b^3\right)2}{3(c-b)(c-a)} + \frac{2c\left(c^2-b^2\right)}{2(c-b)(c-a)} = \\
&= \frac{2\left(b^2+ab+a^2\right)}{3(c-a)} - \frac{ab+a^2}{c-a} - \frac{2\left(c^2+bc+b^2\right)}{3(c-a)} + \frac{c^2+bc}{c-a} = \\
&= \frac{1}{3(c-a)}\left(2b^2+2ab+2a^2-3ab+3a^2-2c^2+2bc+2b^2+3c^2+3bc\right) = \\
&= \frac{\left(c^2+bc\right)-\left(ab+a^2\right)}{3(c-a)} = \frac{c^2-a^2+bc-ab}{3(c-a)} = \frac{a+b+c}{3}.
\end{aligned} \quad (22)$$

Thus, the first of the tasks set has been solved. Let's move on to the second problem.

The technique considered above for each pair (i, j) of states specifies the technology for the appropriate processing of the results of observations of the duration of stay in state i before transition to j. As a result of this processing, let's obtain a matrix of estimates $\bar{T} = \left(\bar{T}_{ij}\right)$ of the expected values of the duration of stay in each state before leaving for each of the possible states. Let's use this matrix to calculate the expected values for the system to stay in each of their states before leaving for any other states according to the formula:

$$\bar{T} = \sum_{j=1}^{n} \bar{T}_{ij}, \; i = 1,2,...,n. \quad (23)$$

Let's now calculate the estimates of the transition probabilities. For a pair (i, j) of states, the number of corresponding transitions S_{ij} is fixed. The resulting set $\{S_{ij}\}$ is used to calculate estimates of transition probabilities:

$$\widehat{P}_{ij} = \frac{S_{ij}}{\sum_{j=1}^{n} S_{ij}}, \ i = 1,2,...,n, \ j = 1,2,...,n. \quad (24)$$

In this case, of course, the equality:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \widehat{P}_{ij} = 1.$$

Using the resulting matrix $P = (P_{ij})$, let's calculate the distribution of the final state probabilities for the nested Markov chain:

$$\pi = \pi P.$$

Let's finally calculate the required stationary probability distribution of the states of the system. This distribution, using $\pi = (\pi_1, \pi_2, ..., \pi_n)$, $\overline{T} = (\overline{T}_1, \overline{T}_2, ..., \overline{T}_n)$, is found by the formulas:

$$P_i = \frac{\pi_i \overline{T}_i}{\sum_{i=1}^{n} \pi_i \overline{T}_i}, \ i = 1,2,...,n. \quad (25)$$

The resulting distribution is a set of point estimates of the probabilities of the system states that satisfy the normalization condition. Thus, the second task of the study is solved.

Note that the simplicity of the model used to describe the core element of the technique – the transition mechanism in the form of an (L–R)-type membership function – does not compensate for its fundamental drawback – the obvious low adequacy. A natural effective possibility to improve the adequacy of the model is to take into account the real variability of the operating conditions of the systems. The resulting inhomogeneity of the initial data can be reliably detected and adequately described using the division of the process of processing the initial data into a certain number of independent sessions. Let at the same time, as a result of processing the sequence $S = 1, 2, ..., r$ of observation sessions using the proposed methodology for a pair of states (i, j), the corresponding sequence of membership functions $\mu(T_{ij})$ of the fuzzy duration of the system's stay in i before transition to j. This sequence is described by the sets $\Phi_m(T_{ij})$. Let's choose a set $\Phi_m(T_{ij})$. It is clear that it is natural to consider each number from this set $\Phi_m(T_{ij}) = \left\{m_{T_{ij}}^{(1)}, m_{T_{ij}}^{(2)}, ..., m_{T_{ij}}^{(r)}\right\}$ as the result of observing the modal value of the fuzzy number T_{ij} in the corresponding session. The totality of these observations can be used to determine the membership function of this fuzzy number T_{ij} with the implementation of the proposed methodology (4) (17). As a result of processing the set $\Phi_m(T_{ij})$, the set $<m_m, \alpha_m, \beta_m>$ will be obtained. Here, for simplicity, the indices (i, j) corresponding to the selected pair of states are omitted. The above reasoning regarding the modal values of the observed fuzzy numbers T_{ij} can be repeated without changes for the values of the left and right fuzziness coefficients of the numbers T_{ij} after which there are the sets:

$$\Phi_\alpha(T_{ij}) = \left\{\alpha_{T_{ij}}^{(1)}, \alpha_{T_{ij}}^{(2)}, ..., \alpha_{T_{ij}}^{(r)}\right\}, \ \Phi_\beta(T_{ij}) = \left\{\beta_{T_{ij}}^{(1)}, \beta_{T_{ij}}^{(2)}, ..., \beta_{T_{ij}}^{(r)}\right\}.$$

As a result of processing these sets, let's obtain the sets $<m_\alpha, \alpha_\alpha, \beta_\alpha>$, $<m_\beta, \alpha_\beta, \beta_\beta>$.

Thus, each fuzzy number T_{ij}, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, n$, the parameters of the membership function of which are themselves described fuzzy, is converted into the corresponding bi-fuzzy number. Let's consider the method of practical use of the obtained much more informative description of the membership functions of fuzzy durations of the system's stay in each of its possible states.

Let's formulate the corresponding problem. So, as a result of preliminary processing of the entire set of observations of the process of the system functioning, for each pair of states (i, j), the membership function $\mu_m(T_{ij})$ of a fuzzy number (L–R)-type, represented by the set $\left\langle m_{T_{ij}}, \alpha_{T_{ij}}, \beta_{T_{ij}} \right\rangle$. Further processing of the initial array of observations divided into groups provides the possibility of obtaining a more accurate description of the duration of the system's stay in state i before transition to state j. At the same time, the fuzzy parameters $\langle m_{T_{ij}}, \alpha_{T_{ij}}, \beta_{T_{ij}} \rangle$ of the membership function $\mu_m(T_{ij})$ of the fuzzy number T_{ij}, in accordance with the technology of their representation in the form of (L–R)-type numbers, are described by the sets $\langle m_{m_{T_{ij}}}, \alpha_{m_{T_{ij}}}, \beta_{m_{T_{ij}}} \rangle$, $\langle m_{\alpha_{T_{ij}}}, \alpha_{\alpha_{T_{ij}}}, \beta_{\alpha_{T_{ij}}} \rangle$, $\langle m_{\beta_{T_{ij}}}, \alpha_{\beta_{T_{ij}}}, \beta_{\beta_{T_{ij}}} \rangle$. Based on this more accurate description of the fuzzy numbers T_{ij}, let's obtain a more accurate representation of the required probability distribution of the system states. In this case, relations (23), (25), which were used earlier in calculating this distribution, must be modernized taking into account the bi-fuzzy character of the numbers T_{ij}.

Let's introduce two bi-fuzzy numbers x_1 and x_2 with membership functions:

$$\mu(x_1) = \langle m_1, \alpha_1, \beta_1 \rangle, \ \mu(x_2) = \langle m_2, \alpha_2, \beta_2 \rangle. \quad (26)$$

Let's describe the fuzzy parameters of these membership functions with our own membership functions:

$$\mu(m_1) = \langle m_{m_1}, \alpha_{m_1}, \beta_{m_1} \rangle; \ \mu(\alpha_1) = \langle m_{\alpha_1}, \alpha_{\alpha_1}, \beta_{\alpha_1} \rangle; \ \mu(\beta_1) = \langle m_{\beta_1}, \alpha_{\beta_1}, \beta_{\beta_1} \rangle; \quad (27)$$

$$\mu(m_2) = \langle m_{m_2}, \alpha_{m_2}, \beta_{m_2} \rangle; \ \mu(\alpha_2) = \langle m_{\alpha_2}, \alpha_{\alpha_2}, \beta_{\alpha_2} \rangle; \ \mu(\beta_2) = \langle m_{\beta_2}, \alpha_{\beta_2}, \beta_{\beta_2} \rangle. \quad (28)$$

The rules for performing operations on bi-fuzzy numbers are based on the corresponding rules for fuzzy numbers [10]. These rules for a pair of fuzzy numbers x_1 and x_2, represented by (26), have the following form.

$$\mu(x_1 + x_2) = \langle m, \alpha, \beta \rangle, \ m = m_1 + m_2, \ \alpha = \alpha_1 + \alpha_2, \ \beta = \beta_1 + \beta_2; \quad (29)$$

$$\mu(x_1 - x_2) = \langle m, \alpha, \beta \rangle, \ m = m_1 - m_2, \ \alpha = \alpha_1 - \alpha_2, \ \beta = \beta_1 - \beta_2; \quad (30)$$

$$\mu(x_1 x_2) = \langle m, \alpha, \beta \rangle, \ m = m_1 m_2, \ \alpha = m_1\alpha_2 + m_2\alpha_1 - \alpha_1\alpha_2, \ \beta = m_1\beta_2 + m_2\beta_1 + \beta_1\beta_2; \quad (31)$$

$$\mu(x_1 / x_2) = \langle m, \alpha, \beta \rangle, \ m = m_1 / m_2, \ \alpha = \frac{m_2\alpha_1 + m_1\beta_2}{m_2(m_2 + \beta_2)}, \ \beta = \frac{m_1\alpha_2 + m_2\beta_1}{m_2(m_2 - \alpha_2)}. \quad (32)$$

To implement relations (23) and (25) with bi-fuzzy arguments, it is necessary to perform the operations of adding bi-fuzzy numbers, multiplying a bi-fuzzy number by a scalar, and dividing bi-fuzzy numbers. Let's use relations (29) (32) to describe the necessary operations on bi-fuzzy numbers.

Addition. In accordance with (29), taking into account (27), (28), let's find the parameters of the membership functions of numbers $m_1 + m_2$, $\alpha_1 + \alpha_2$, $\beta_1 + \beta_2$.

$$m = m_1 + m_2 = \langle m_{m_1} + m_{m_2}, \alpha_{m_1} + \alpha_{m_2}, \beta_{m_1} + \beta_{m_2} \rangle,$$

$$\alpha = \alpha_1 + \alpha_2 = \langle m_{\alpha_1} + m_{\alpha_2}, \alpha_{\alpha_1} + \alpha_{\alpha_2}, \beta_{\alpha_1} + \beta_{\alpha_2} \rangle,$$

$$\beta = \beta_1 + \beta_2 = \langle m_{\beta_1} + m_{\beta_2}, \alpha_{\beta_1} + \alpha_{\beta_2}, \beta_{\beta_1} + \beta_{\beta_2} \rangle. \quad (33)$$

Multiplication. In accordance with (31), the parameters of the membership function of the fuzzy number $x_1 x_2$, taking into account the fuzziness of the parameters of the membership function of each of the factors determined by (27), (28), is calculated by sequentially calculating the membership functions of the following products of fuzzy numbers $m_1 m_2$, $m_{12}\alpha_1$, $m_1\alpha_1$, $\alpha_1\alpha_2$, $m_1\beta_2$, $m_2\beta_1$, $\beta_1\beta_2$. The corresponding relations have the form [16, 17]:

$$m_1 m_2 = \langle m_{m_1} m_{m_2}; m_{m_1}\alpha_{m_2} + m_{m_2}\alpha_{m_1} - \alpha_{m_1}\alpha_{m_2}; m_{m_1}\beta_{m_2} + m_{m_2}\beta_{m_1} + \beta_{m_1}\beta_{m_2} \rangle;$$

$$m_1 \alpha_2 = \langle m_{m_1} m_{\alpha_2}; m_{m_1}\alpha_{\alpha_2} + m_{\alpha_2}\alpha_{m_1} - \alpha_{m_1}\alpha_{\alpha_2}; m_{m_1}\beta_{\alpha_2} + m_{\alpha_2}\beta_{m_1} + \beta_{m_1}\beta_{\alpha_2} \rangle;$$

$$m_2\alpha_1=\left\langle m_{m_2}m_{\alpha_1};m_{m_2}\alpha_{\alpha_1}+m_{\alpha_1}\alpha_{m_2}-\alpha_{m_2}\alpha_{\alpha_1};m_{m_2}\beta_{\alpha_1}+m_{\alpha_1}\beta_{m_2}+\beta_{m_2}\beta_{\alpha_1}\right\rangle;$$

$$\alpha_1\alpha_2=\left\langle m_{\alpha_1}m_{\alpha_2};m_{\alpha_1}\alpha_{\alpha_2}+m_{\alpha_2}\alpha_{\alpha_1}-\alpha_{\alpha_1}\alpha_{\alpha_2};m_{\alpha_1}\beta_{\alpha_2}+m_{\alpha_2}\beta_{\alpha_1}+\beta_{\alpha_1}\beta_{\alpha_2}\right\rangle;$$

$$m_1\beta_2=\left\langle m_{m_1}m_{\beta_2};m_{m_1}\alpha_{\beta_2}+m_{\beta_2}\alpha_{m_1}-\alpha_{m_1}\alpha_{\beta_2};m_{m_1}\beta_{\beta_2}+m_{\beta_2}\beta_{m_1}+\beta_{m_1}\beta_{\beta_2}\right\rangle;$$

$$m_2\beta_1=\left\langle m_{m_2}m_{\beta_1};m_{m_2}\alpha_{\beta_1}+m_{\beta_1}\alpha_{m_2}-\alpha_{m_2}\alpha_{\beta_1};m_{m_2}\beta_{\beta_1}+m_{\beta_1}\beta_{m_2}+\beta_{m_2}\beta_{\beta_1}\right\rangle;$$

$$\beta_1\beta_2=\left\langle m_{\beta_1}m_{\beta_2};m_{\beta_1}\alpha_{\beta_2}+m_{\beta_2}\alpha_{\beta_1}-\alpha_{\beta_1}\alpha_{\beta_2};m_{\beta_1}\beta_{\beta_2}+m_{\beta_2}\beta_{\beta_1}+\beta_{\beta_1}\beta_{\beta_2}\right\rangle.$$

If one of the factors is a scalar, then these formulas are simplified. Let x_2 be a scalar described in terms of fuzzy mathematics as follows:

$$\mu(x_2)=\langle m_2,\alpha_2,\beta_2\rangle=\langle m_2,0,0\rangle.$$

At the same time, of course,

$$m_2=m_{m_2},\ \alpha_{m_2}=\beta_{m_2}=m_{\alpha_2}=\alpha_{\alpha_2}=\beta_{\alpha_2}=m_{\beta_2}=\alpha_{\beta_2}=\beta_{\beta_2}=0.$$

Then:

$$m_1m_2=\left\langle m_{m_1}m_{m_2},\ m_{m_2}\alpha_{m_1},\ m_{m_2}\beta_{m_1}\right\rangle,$$

$$m_2\alpha_1=\left\langle m_{m_2}m_{\alpha_1},\ m_{m_2}\alpha_{\alpha_1},\ m_{m_2}\beta_{\alpha_1}\right\rangle,$$

$$m_2\beta_1=\left\langle m_{m_2}m_{\beta_1},\ m_{m_2}\alpha_{\beta_1},\ m_{m_2}\beta_{\beta_1}\right\rangle. \quad (34)$$

Division. In accordance with (32), the parameters of the membership function of the fuzzy number x_1/x_2, taking into account the fuzziness of the operands, the membership functions of which are given by (27), (28), are determined by the results of calculating the membership functions of the following fuzzy numbers: m_1/m_2, $m_1\alpha_2$, $m_2\alpha_1$, $m_1\beta_2$, $m_2\beta_1$, $m_1\alpha_2+m_2\beta_1$, $m_2(m_2-\alpha_2)$,

$$\frac{m_2\alpha_1+m_1\beta_2}{m_2(m_2+\beta_2)},\ \frac{m_1\alpha_2+m_2\beta_1}{m_2(m_2-\alpha_2)}.$$

Let's obtain the corresponding ratios. The formula for calculating the parameters of the membership function of a fuzzy number m_1/m_2 can easily be written if the modal values of the fuzzy components are substituted into the general relation (32). In doing so, let's obtain:

$$\frac{m_1}{m_2}=\left\langle\frac{m_{m_1}}{m_{m_2}};\frac{m_{m_2}\cdot\alpha_{m_1}+m_{m_1}\cdot\beta_{m_2}}{m_{m_2}\left(m_{m_2}+\beta_{m_2}\right)};\frac{m_{m_1}\cdot\alpha_{m_2}+m_{m_2}\cdot\beta_{m_1}}{m_{m_2}\left(m_{m_2}-\alpha_{m_2}\right)}\right\rangle.$$

Relationships for calculating the parameters of the membership function of fuzzy numbers $m_1\alpha_2$, $m_1\alpha_2$, $m_2\alpha_1$, $m_1\beta_2$, $m_2\beta_1$ were obtained earlier in (13), (14), (16), (17). Next:

$$m_{m_2}\beta_{\alpha_1}+m_{\alpha_1}\beta_{m_2}+m_{\alpha_1}\beta_{m_2}+\beta_{m_2}\beta_{\alpha_1}+m_{m_1}\beta_{\beta_2}+m_{\beta_2}\beta_{m_1}+\beta_{m_1}\beta_{\beta_2}>$$

$$m_2+\beta_2=<m_{m_2}+m_{\beta_2};\ \alpha_{m_2}+\alpha_{\beta_2};\ \beta_{m_2}\beta_{\beta_2}>;$$

$$m_2(m_2+\beta_2)=<m_{m_2}\left(m_{m_2}+m_{\beta_2}\right);\ m_{m_1}\left(\alpha_{m_2}+\alpha_{\beta_2}\right)+\left(m_{m_2}+m_{\beta_2}\right)\alpha_{m_2}-\beta_{m_2}\left(\beta_{m_2}+\beta_{\beta_2}\right);$$

$$m_{m_1}\left(\beta_{m_2}+\beta_{\beta_2}\right)+\left(m_{m_2}+m_{\beta_2}\right)\beta_{m_2}+\beta_{m_2}\left(\beta_{m_2}+\beta_{\beta_2}\right)>.$$

$$m_1\alpha_2+m_2\beta_1\le m_{m_1}m_{\alpha_2}+m_{m_2}m_{\beta_1};$$

$$m_{m_1}\alpha_{\alpha_2}+m_{\alpha_2}\alpha_{m_1}-\alpha_{m_1}\alpha_{\alpha_2}+m_{m_2}\alpha_{\beta_1}+m_{\beta_1}\alpha_{m_2}-\alpha_{m_2}\alpha_{\beta_1};$$

$$m_{m_1}\beta_{\alpha_2}+m_{m_2}\beta_{m_1}+\beta_{m_1}\beta_{\alpha_2}+m_{m_2}\beta_{\beta_1}+m_{\beta_1}\beta_{m_2}+\beta_{m_2}\beta_{\beta_1}>.$$

$$m_2-\alpha_2=<m_{m_2}-m_{\alpha_2};\ \alpha_{m_2}+\beta_{\alpha_2};\ \beta_{m_2}+\alpha_{\alpha_2}>;$$

$$m_2(m_2-\alpha_2)=<m_{m_2}\left(m_{m_2}-m_{\alpha_2}\right);$$

$$m_{m_2}\left(\alpha_{m_2}+\beta_{\alpha_2}\right)+\left(m_{m_2}-m_{\alpha_2}\right)\alpha_{m_2}-\alpha_{m_2}\left(\alpha_{m_2}+\beta_{\alpha_2}\right);$$

$$m_{m_2}\left(\beta_{m_2}+\alpha_{\alpha_2}\right)+\left(m_{m_2}-m_{\alpha_2}\right)\beta_{m_2}+\beta_{m_2}\left(\beta_{m_2}+\beta_{\beta_2}\right)>.$$

The obtained relations define the formulas that determine the numerator and denominator of the expressions for the fuzziness coefficients in (32). Now there is a possibility of direct calculation using analogues of formulas (23) and (25) of the distribution of the final probabilities of the system, taking into account the binarity of the initial data.

It is clear that the introduction of uncertainty models based on second-order fuzzy numbers into the system analysis procedure increases the adequacy of these models. Consider the simplest example that confirms this conclusion.

Let a semi-Markov recoverable system have two possible states:

– E_0 – the system is working;

– E_1 – the system is recovering from a failure.

Let's analyze this system using a probabilistic model of its functioning. Let's introduce a set of independent distribution functions for the duration of the system's stay in its possible states before leaving:

– $Q_{01}(t) = F_0(t)$ – distribution law of the interval between failures,

– $Q_{10}(t) = F_1(t)$ – the law of distribution of recovery time.

Wherein $P_{01}(t) = Q_{01}(t)$ – the probability of the system transition from state E_0 to state E_1 in time t, $P_{01}(\infty) = 1$.

$P_{10}(t) = Q_{10}(t)$ – the probability of the system transition from state E_1 to state E_0 in time t, $P_{10}(\infty) = 1$.

Then $P=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ – the transition probability matrix for the nested Markov chain describing the behavior of the system.

Let's introduce further $F_{01}(t) = P_{01}(t) = F_0(t)$ – the distribution function of the duration of stay in E_0 before leaving for E_1, $f_{01}(t)=dF_{01}(t)/dt$ – the distribution density of the duration of stay in E_0 before leaving for E_1.

$F_{10}(t) = P_{10}(t) = F_1(t)$ – the distribution function of the duration of stay in E_1 before leaving for E_0, $f_{10}(t)=dF_{10}(t)/dt$ – the distribution density of the duration of stay in E_1 before leaving for E_0.

To analyze the system, let's use the mathematical apparatus of interval-transitional probabilities. For this purpose, let's introduce.

$\varphi_{ij}(t)$ – the conditional probability that at time t the system is in state j if at time $t = 0$ it was in state i. This probability is called interval transition. The set of interval transition probabilities satisfies the system of integral equations.

$$\varphi_{ij}(t)=\delta_{ij}\psi_i(t)+\sum_{\substack{k\in E\\ k\neq i}}P_{ik}\int_0^t f_{ik}(\tau)\varphi_{ik}(t-\tau)\mathrm{d}\tau, \tag{35}$$

E – set of possible states:

$$\psi_i(t)=1-F_i(t),\ F_i(t)=\sum_{\substack{j\in E\\ j\neq i}}P_{ij}F_{ij}(t). \tag{36}$$

The system of equations (35) is solved using the Laplace transform. In doing so, let's obtain:

$$\varphi^*_{ij}(s)=\delta_{ij}\psi^*_i(s)+\sum_{\substack{k\in E\\ k\neq i}}P_{ik}f^*_{ik}(s)\varphi^*_{ik}(s). \tag{37}$$

Let's introduce matrices $\varphi(t) = (\varphi_{ij}(t))$, $P = (p_{ij}(t))$, $f(t) = (f_{ij}(t))$, diagonal matrices $W(t) = (\delta_{ij} f_i(t))$, $F_i(t) = (\delta_{ij} F_i(t))$, $\psi(t) = (\delta_{ij}\psi(t))$ and their corresponding Laplace transform matrices. Using these matrices, let's obtain a compact representation of the system of equations (37). There is:

$$\varphi^*(s) = \psi^*(s) + \left[Pof^*(s)\right]\varphi^*(s). \quad (38)$$

Here the record of the form $C = AoB$ determines the specific nature of the execution of the operation of matrix multiplication, according to which $c_{ij} = a_{ij}b_{ij}$. Equation (38) is transformed into the form:

$$\varphi^*(s) = \left[I - Pof^*(s)\right]^{-1}\psi^*(s). \quad (39)$$

Further, since:

$$\psi^*(s) = \int_0^\infty e^{-st}\left[1 - \int_0^t f_i(\tau)\mathrm{d}\tau\right]\mathrm{d}t = -\frac{e^{-st}}{s}\bigg|_0^\infty - \frac{f_i^*(s)}{s} = \frac{1}{s}\left(1 - f_i^*(s)\right), \quad (40)$$

then relation (39) is written as follows:

$$\varphi^*(s) = \left[I - Pof(s)\right]^{-1}\frac{1}{s}(I - W), \quad (41)$$

where, according to (40),

$$\psi^*(s) = \frac{1}{s}(I - W).$$

Let's use the obtained general relations to solve the specific problem of analyzing the system being restored formulated above. For this system:

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ f_{01}(t) = f_0(t),\ f_{10}(t) = f_1(t),$$

$$f(t) = \begin{pmatrix} 0 & f_0(t) \\ f_1(t) & 0 \end{pmatrix},\ f^*(s) = \begin{pmatrix} 0 & f_0(s) \\ f_1(s) & 0 \end{pmatrix},$$

$$\psi^*(s) = \begin{pmatrix} \psi(s) & 0 \\ 0 & \psi(s) \end{pmatrix},\ f^*(s) = \begin{pmatrix} 1 - f_0(s) & 0 \\ 0 & 1 - f_1(s) \end{pmatrix}.$$

Then relation (41) is transformed as follows:

$$\varphi^*(s) = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} o \begin{pmatrix} 0 & f_0(s) \\ f_1(s) & 0 \end{pmatrix}\right]^{-1}\begin{pmatrix} 1 - f_0(s) & 0 \\ 0 & 1 - f_1(s) \end{pmatrix}\frac{1}{s} =$$

$$= \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & f_0(s) \\ f_1(s) & 0 \end{pmatrix}\right]^{-1}\begin{pmatrix} 1 - f_0(s) & 0 \\ 0 & 1 - f_1(s) \end{pmatrix}\frac{1}{s} =$$

$$= \begin{pmatrix} 1 & -f_0(s) \\ -f_1(s) & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 - f_0(s) & 0 \\ 0 & 1 - f_1(s) \end{pmatrix}\frac{1}{s} =$$

$$= \frac{1}{s\left(1 - f_0(s)f_1(s)\right)}\begin{pmatrix} 1 & f_0(s) \\ f_1(s) & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 - f_0(s) & 0 \\ 0 & 1 - f_1(s) \end{pmatrix} =$$

$$= \frac{1}{s\left(1 - f_0(s)f_1(s)\right)}\begin{pmatrix} 1 - f_0(s) & f_0(s)\left(1 - f_1(s)\right) \\ f_1(s)\left(1 - f_0(s)\right) & 1 - f_1(s) \end{pmatrix}. \quad (42)$$

Relation (42) represents the solution of the problem of system analysis in terms of the Laplace transform. The desired probability distribution of the system can be obtained using the inverse Laplace transform for specific descriptions of the distribution density of the duration of stay in each of the states of the system before leaving.

Let, for example, in the simplest possible particular case $f_0(t) = \lambda e^{-\lambda t}$, $f_1(t) = \mu e^{-\mu t}$. Then,

$$f_0(s)=\frac{\lambda}{s+\lambda},\ f_1(s)=\frac{\mu}{s+\mu}.$$

Wherein,

$$\varphi^*(s)=\frac{1}{s\left(1-\frac{\lambda\mu}{(s+\lambda)(s+\mu)}\right)}\begin{pmatrix}1-\frac{\lambda}{s+\lambda} & \frac{\lambda}{s+\lambda}\left(1-\frac{\mu}{s+\mu}\right)\\ \frac{\mu}{s+\mu}\left(1-\frac{\lambda}{s+\lambda}\right) & 1-\frac{\mu}{s+\mu}\end{pmatrix}=$$

$$=\frac{(s+\lambda)(s+\mu)}{s\left(s^2+s(\lambda+\mu)\right)}\begin{pmatrix}\frac{s}{s+\lambda} & \frac{s\lambda}{(s+\lambda)(s+\mu)}\\ \frac{s\mu}{(s+\lambda)(s-\mu)} & \frac{s}{s+\mu}\end{pmatrix}=\frac{1}{s(s+\lambda+\mu)}\begin{pmatrix}s+\mu & \lambda\\ \mu & s+\lambda\end{pmatrix}.$$

From here:

$$\varphi_{00}(s)=\frac{1}{s(s+\lambda+\mu)}=\frac{\mu}{s(\lambda+\mu)}+\frac{\lambda}{(\lambda+\mu)(s+\lambda+\mu)}.$$

Performing the inverse Laplace transform, let's obtain:

$$\varphi_{00}(s)=\frac{\mu}{\lambda+\mu}+\frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t}. \tag{43}$$

Similar actions with other elements of the matrix $\Phi(s)$ lead to the following results:

$$\varphi_{01}(s)=\frac{\lambda}{\lambda+\mu}\left(1-e^{-(\lambda+\mu)t}\right), \tag{44}$$

$$\varphi_{10}(s)=\frac{\mu}{\lambda+\mu}\left(1-e^{-(\lambda+\mu)t}\right), \tag{45}$$

$$\varphi_{00}(s)=\frac{\lambda}{\lambda+\mu}+\frac{\lambda}{\lambda+\mu}\frac{\mu}{\lambda+\mu}e^{-(\lambda+\mu)t}. \tag{46}$$

Let's now solve the problem of analyzing the same restoration system for the case when the analytical descriptions of the distribution laws for the duration of stay in each of the states are not clearly defined. Let the membership function of the fuzzy value of the duration T_0 of stay in the state E_0 before the transition to E_1 has the form:

$$\mu(T_0)=\begin{cases}0,\ T_0\le a_0,\\ \frac{T_0-a_0}{b_0-a_0},\ a_0\le T_0\le b_0,\\ \frac{c_0-T_0}{c_0-b_0},\ b_0\le T_0\le c_0,\\ 0,\ T_0>c_0.\end{cases} \tag{47}$$

In this case, in accordance with (22), the average duration of stay in E_0 before leaving for E_1 is equal to:

$$\bar{T}_0=\frac{a_0+b_0+c_0}{3}. \tag{48}$$

Similarly, let's introduce the membership function of the fuzzy value of the duration T_1 of stay in E_1 before the transition E_0:

$$\mu(T_1)=\begin{cases}0, T_1\le a_1,\\ \dfrac{T_1-a_1}{b_1-a_1}, a_1\le T_1\le b_1,\\ \dfrac{c_1-T_1}{c_1-b_1}, b_1\le T_1\le c_1,\\ 0, T_1>c_1.\end{cases} \quad (49)$$

At the same time, the average duration of stay in E_1 before leaving for E_0 is equal to:

$$\overline{T}_1=\frac{a_1+b_1+c_1}{3}. \quad (50)$$

Further, the matrix of transition probabilities in the system under consideration has the same form as before $P=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$.

Let's introduce the matrix system of equations:

$$(\pi_0\pi_1)=(\pi_0\pi_1)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

regarding the probabilities of the states of the system.

This system has a simple solution: $\pi_0=\pi_1=1/2$.

Now, using (25), let's obtain the distribution of the final probabilities of the system states:

$$P_0=\frac{\pi_0\overline{T}_0}{\pi_0T_0+\pi_1T_1},\ P_1=\frac{\pi_1\overline{T}_1}{\pi_0T_0+\pi_1T_1}. \quad (51)$$

It is clear that the solution obtained, in view of the low accuracy of models (47), (49), should be considered the first approximation to the description of the system functioning process. Therefore, in accordance with the proposed methodology, let's move on to a more adequate representation of this process in terms of bi-fuzzy mathematics.

Let, as a result of processing the initial array of observations divided into groups, the membership functions of the parameters of the membership functions (47), (49), given by the tuples:

$$a_0=\langle m_{a_0},\alpha_{a_0},\beta_{a_0}\rangle,\ b_0=\langle m_{b_0},\alpha_{b_0},\beta_{b_0}\rangle,\ c_0=\langle m_{c_0},\alpha_{c_0},\beta_{c_0}\rangle,$$

$$a_1=\langle m_{a_1},\alpha_{a_1},\beta_{a_1}\rangle,\ b_1=\langle m_{b_1},\alpha_{b_1},\beta_{b_1}\rangle,\ c_1=\langle m_{c_1},\alpha_{c_1},\beta_{c_1}\rangle. \quad (52)$$

The average values of fuzzy numbers a_0, b_0, c_0, a_1, b_1, c_1 are determined by the relations [16]:

$$\overline{a}_0=\frac{(m_{a_0}-\alpha_{a_0})+m_{a_0}+(m_{a_0}+\beta_{a_0})}{3}=m_{a_0}+\frac{\beta_{a_0}-\alpha_{a_0}}{3},$$

$$\overline{b}_0=\frac{(m_{b_0}-\alpha_{b_0})+m_{b_0}+(m_{b_0}+\beta_{b_0})}{3}=m_{b_0}+\frac{\beta_{b_0}-\alpha_{b_0}}{3},$$

$$\overline{c}_0=\frac{(m_{c_0}-\alpha_{c_0})+m_{c_0}+(m_{c_0}+\beta_{c_0})}{3}=m_{c_0}+\frac{\beta_{c_0}-\alpha_{c_0}}{3},$$

$$\overline{a}_1=\frac{(m_{a_1}-\alpha_{a_1})+m_{a_1}+(m_{a_1}+\beta_{a_1})}{3}=m_{a_1}+\frac{\beta_{a_1}-\alpha_{a_1}}{3},$$

$$\overline{b}_1=\frac{(m_{b_1}-\alpha_{b_1})+m_{b_1}+(m_{b_1}+\beta_{b_1})}{3}=m_{b_1}+\frac{\beta_{b_1}-\alpha_{b_1}}{3},$$

$$\overline{c}_1=\frac{(m_{c_1}-\alpha_{c_1})+m_{c_1}+(m_{c_1}+\beta_{c_1})}{3}=m_{c_1}+\frac{\beta_{c_1}-\alpha_{c_1}}{3}. \quad (53)$$

Let's substitute the obtained estimates of the average values of the parameters of the membership functions of the bi-fuzzy numbers T_0 and T_1 into (48) and (50) to calculate the expected values $\overline{T}_0$ and $\overline{T}_1$ of these numbers as well. The final operation is the substitution of and in (51) to obtain the required probability distribution of the system states.

Let's illustrate the practical expediency of applying the described technique with a numerical example.

Let's set fuzzy numbers T_0 and T_1:

$$T_0 = \langle m_{T_0}, \alpha_{T_0}, \beta_{T_0} \rangle = \langle 120,60,30 \rangle,\ T_1 = \langle m_{T_1}, \alpha_{T_1}, \beta_{T_1} \rangle = \langle 15,10,55 \rangle.$$

Using (22), let's obtain $\overline{T}_0 = 110,\ \overline{T}_1 = 30.$

Next, using formulas (23), let's calculate the probabilities of the states:

$$P_1 = \frac{\frac{1}{2}110}{\frac{1}{2}110 + \frac{1}{2}30} = 0.785;\ P_2 = \frac{\frac{1}{2}30}{\frac{1}{2}110 + \frac{1}{2}30} = 0.215.$$

Let, now, based on the results of refining calculations, descriptions of fuzzy parameters of membership functions (47), (49) are obtained in the form (52):

$$a_0 = \langle 110,50,10 \rangle,\ b_0 = \langle 120,0,0 \rangle,\ c_0 = \langle 145,5,5 \rangle,$$

$$a_1 = \langle 7.5,2.5,2.5 \rangle,\ b_1 = \langle 15,0,0 \rangle,\ c_1 = \langle 30,10,40 \rangle.$$

Using (53), let's calculate the average values of fuzzy numbers a_0, b_0, c_0, a_1, b_1, c_1. In doing so:

$$\overline{a}_0 = 96.6,\ \overline{b}_0 = 120,\ \overline{c}_0 = 145,$$

$$\overline{a}_1 = 7.5,\ \overline{b}_1 = 15,\ \overline{c}_1 = 40.$$

Then, according to (22):

$$\overline{T}_1 = \frac{96.6 + 120 + 145}{3} = 120.3,\ \overline{T}_2 = \frac{7.5 + 15 + 40}{3} = 20.8.$$

Now the calculation by formulas (23) gives the distribution of the final probabilities of the system states.

$$P_1 = \frac{120.3}{141.1} = 0.85;\ P_2 = \frac{20.8}{141.1} = 0.15.$$

The obtained refined probability distribution of the system states differs markedly from the previous one, thus naturally confirming the expediency of searching and using the most adequate models.

Thus, the urgent need to adequately take into account the real uncertainty in the estimates of the values of controlled parameters of dynamic systems led to the need to develop an appropriate mathematical apparatus. Adequate formulation of the problems of analysis of such systems determined the direction and purpose of research. It is clear that in specific problems of practical analysis of the efficiency of real systems, the necessary initial data on the parameters of systems can be obtained only through proper statistical processing of their direct measurements. These data are further correctly used to restore the membership functions of the fuzzy parameters of the analyzed system. Analytical descriptions of these functions are obtained by the maximum likelihood method and are the basis of the proposed method for analyzing fuzzy semi-Markov systems. This method radically improves the well-known interval method for solving the problem, the possibilities of which are limited to obtaining only pessimistic and optimistic estimates of the efficiency of real systems. The obtained method offers a new toolkit for solving the problem of analysis of semi-Markov systems, in the case when the parameters of membership functions of fuzzy measurements are themselves fuzzy. The possibility of calculating the membership functions of the bi-fuzzy numbers that arise in this case is considered. This method significantly increases the adequacy of solving practical problems of the analysis of semi-Markov systems, filling the existing gap in the general theory of the study of complex systems under conditions of real hierarchical uncertainty.

The direction of further research is related to the development of a method for analyzing systems in an even more complex situation, when, as a result of the limited initial data, correct statistical estimation is provided only when calculating the expected value and variation of the estimated parameters. The solution of the problem in this case is possible using the theory of continual linear programming.

4. Conclusions

The semi-Markov model of functioning of complex systems is considered. A technology for the analysis of such systems is proposed for the following standard models for setting the initial data: probabilistic, fuzzy, and bi-fuzzy.

A technique for estimating the parameters of the membership functions of the parameters of the system model based on the results of processing an array of fuzzy initial data has been developed. At the same time, a situation typical for practice is considered, when these data have hierarchical uncertainty, that is, they are bi-fuzzy. Using the proposed rules for performing operations on a set of such numbers, a method for calculating the final probability distribution of system states is obtained. A real possibility of a significant increase in the adequacy of traditional fuzzy models of systems functioning due to the use of a binomial description of the initial data is shown.

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