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DETERMINATION OF THE GREEN FUNCTION OF A PULSED ACOUSTIC SOURCE IN A UNIFORM HOMOGENEOUS FLOW WITH AN ARBITRARY MACH NUMBER

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Abstract

The wave field created by a pulsed point source of sound in a uniform homogeneous flow with an arbitrary value of the Mach number is theoretically studied. The aim of research is to obtain an analytical dependence of the sound field on physical parameters.

The space-waveguide Fourier expansion of the sound field is used to solve the Cauchy problem for the wave equation in a reference frame moving together with the medium. It is only necessary to transform the spatiotemporal dependence of the source, given in a fixed frame of reference, to a dependence in a moving frame of reference.

The transition to the description of the solution in the frame of reference, relative to which the medium moves at a constant velocity, is made taking into account the main properties of the generalized Dirac δ-function.

Analytical dependences of the sound field on physical parameters are obtained for both subsonic and supersonic flows. A comparison is made with the results of calculations for the case of a pulsed point source moving in a medium at rest. The solution obtained in this work for the case of an impulsive source moving in a medium at rest made it possible to trace its connection with the Green's function for a stationary source in a moving medium. The analytical dependence of the obtained solution for the Green's function makes it possible to write down the explicit form of the «characteristic solution surface», that is, the «wave front». It is shown that the difference between the solutions for subsonic and supersonic motion is characterized by the position of the source relative to the moving wavefront.

The calculation results can be used to describe the sound field created by an arbitrary spatiotemporal distribution of sound sources.

Keywords: Fourier expansion, Cauchy problem, causality principle, properties of the δ-function, delay, wavefront, Mach number.

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1. Introduction

Perturbation of the state of the medium in any small volume causes them to propagate in the medium in the form of acoustic vibrations under the influence of inertia and elasticity forces.

In the presence of motion of the medium, acoustic phenomena become much more complicated in comparison with the case of a motionless medium. Firstly, there is a «demolition» of the area of propagation of sound vibrations, and secondly, there is a scattering of sound waves in inhomogeneous flows. Accordingly, the mathematical apparatus necessary for describing these processes becomes more complicated.

Studies of acoustic phenomena in a moving medium have a long history. Information about the early stage of research is presented in the well-known monograph [1]. The book systematizes the theoretical developments accumulated by that time on the acoustics of a moving medium, a significant part of which is the result of research by the author himself. The propagation of sound in a constant flow, the approximation of geometric acoustics in an inhomogeneous moving medium, the problems of moving sources and receivers of sound, including those placed in a flow of a moving medium, are analyzed.

Later studies by the beginning of the 2000s are detailed in the monograph [2]. The content of the monograph covers a much larger range of studies, in particular, for stratified media, randomly inhomogeneous media, etc.

The state of the problem under study is currently presented to a certain extent in the review article [3]. The review provides information about Green's functions in free space, in a half-space, and in closed spatial domains. By the nature of the time dependence, the reduced Green's functions refer either to the solution of the wave equation or the Helmholtz equation in the case of a monochromatic source. The cases of stationary and moving media are presented.

An analysis of the available publications indicates a small number of studies of the Cauchy problem, which is quite significant in acoustics of moving media, for a pulsed point sound source (Green's function). For example, in [1], an expression was proposed for the Green's function in a uniform flow containing errors, which were noted in [2]. The inconsistency in the mathematical transformations of the wave equation led to an incorrect result. The review [3] presents the solution obtained by the authors of [4] using the fourfold space-time Fourier transform, which is limited by the subsonic velocity of the medium. A similar solution of the same problem [5] by another method is also limited to the case of a subsonic flow. Therefore, the search for a solution to the problem under study for arbitrary values of the Mach number of the moving medium remains an urgent task.

The aim of research is to determine the wave field created by a stationary impulsive sound source in an unlimited moving medium. The solution of such a problem in physics is usually called the Green's function.

The objectives of research are the formulation of mathematical conditions and equations, by solving which it is possible to obtain analytical dependences of the desired solution for the Green's function on physical parameters.

In order to establish the relationship between the sought-for Green's function and the wave field created by a pulsed source when it moves in a medium at rest, the corresponding calculations are included among the research tasks.

2. Materials and methods

The main object of this study is the inhomogeneous wave equation for the velocity potential in an inertial frame of reference (IFR), relative to which a homogeneous medium moves uniformly at a constant velocity $\vec{v}$. The density of volumetric sound sources on the right side of the wave equation is represented as a pulsed point source. The coefficients of the considered equation are constant, which makes it possible to use the Fourier method of separation of variables when solving the equation. With a certain choice of the Cartesian coordinate system, the problem has rotation symmetry around an axis parallel to the vector $\vec{v}$, which greatly facilitates the solution. To separate the process of «carrying» sound from the process of propagation of sound vibrations, the Galilean transformation from a fixed IFR to an IFR moving along with the medium is used. Under the inverse transformation, the resulting solution takes into account the main properties of the generalized Dirac δ-function.

3. Results and discussion

In the frame of reference, relative to which the medium moves at a constant velocity $\vec{v}$, the problem is reduced to solving the wave equation for the velocity potential $\psi(\vec{r},t)$:

$$L\psi(\vec{r},t)=Q(\vec{r},t), \tag{1}$$

under the condition $\psi(\vec{r},t)=0$ for $t<t'$ (principle of causality), where $Q(\vec{r},t)$ the source of mass (the source of volumetric velocity), which is identically equal to zero at $t<t'$, $\vec{r}=(x,y,z)$ – the radius vector of the observation point, t – time, L – wave operator in a homogeneous uniform flow [1, 2]:

$$L=\frac{1}{c_a^2}\left(\frac{\partial}{\partial t}+\vec{v}\nabla\right)^2-\Delta, \tag{2}$$

where c_a – the velocity of sound, ∇ – the Hamilton operator, Δ – the Laplacian.

A point pulsed sound source is located at a point $\vec{r}'$ and acts at time t':

$$Q(\vec{r},t)=\delta(\vec{r}-\vec{r}')\delta(t-t'). \tag{3}$$

In order to simplify the calculations, let's choose the beginning of the Cartesian coordinate system at the point where the source is located and the beginning of the time reference at the moment $t'=0$. Also, the direction of the z-axis is compatible with the direction of the velocity vector $\vec{v}=(0,0,v_z)$.

With this choice, the problem has rotational symmetry around the z-axis, which is extremely important for efficient further calculations.

To move to a reference frame in which $\vec{r}'\neq 0$ and, $t'\neq 0$ in the final results, it is possible to replace $\vec{r}\rightarrow\vec{r}-\vec{r}'$ and $t\rightarrow t-t'$.

With the choice made, the operator L in formula (2) takes the form

$$L=\frac{1}{c_a^2}\cdot\frac{\partial^2}{\partial t^2}+2\frac{M_z}{c_a}\cdot\frac{\partial^2}{\partial t\partial z}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}-\left(1-M_z^2\right)\frac{\partial^2}{\partial z^2}, \tag{4}$$

where the notation $M_z = v_z/c_a$ is introduced The coefficient $\left(1-M_z^2\right)$ in (4) changes sign upon transition to supersonic velocity ($M_z > 1$) of the flow, on the basis of which it was assumed in [1] that the wave equation (1) during such a transition changes its type from hyperbolic to ultrahyperbolic. The erroneousness of this assumption follows from the study of the index of inertia of the quadratic form that determines the characteristic equation [6].

Using a linear transformation of independent variables:

$$\vec{r}=\vec{v}t+\vec{r}_m, t=t_m, \tag{5}$$

operator L is transformed to the form:

$$L=\frac{1}{c_a^2}\cdot\frac{\partial^2}{\partial t_m^2}-\Delta_m. \tag{6}$$

where

$$\Delta_m=\frac{\partial^2}{\partial x_m^2}+\frac{\partial^2}{\partial y_m^2}+\frac{\partial^2}{\partial z_m^2}. \tag{7}$$

The solution $\psi(\vec{r},t)$ for a point source (3) is usually denoted by $G(\vec{r},t)$. In variables $\vec{r}_m$, t_m equation (1) is written as:

$$L_m G_m=\left(\frac{1}{c_a^2}\cdot\frac{\partial^2}{\partial t_m^2}-\Delta_m\right)G_m=Q_m\left(\vec{r}_m,t_m\right), \tag{8}$$

where

$$G_m\equiv G\left(\vec{r}_m+\vec{v}t_m,t_m\right), \tag{9}$$

$$Q_m\left(\vec{r}_m,t_m\right)=\delta\left(\vec{r}_m+\vec{v}t_m\right)\delta\left(t_m\right). \tag{10}$$

It follows from the form (6) that the index of inertia of the quadratic form corresponding to the characteristic surface for equation (8) is equal to unity, and, therefore, the wave equation (1) retains its hyperbolic type under any linear transformations of the independent variables [6].

In its physical meaning, the transformation of variables (5) is a Galileo transformation, which corresponds to the transition from one inertial frame of reference, relative to which the medium moves at a constant velocity $\vec{v}$, to another inertial frame, moving together with the medium. With respect to this system, the medium is stationary and the operator L_m is a wave operator for a stationary medium [1].

Given the choice of $\vec{v}=(0,0,v_z)$ and $\vec{r}'=0$, $t'=0$, the source looks like this:

$$Q_m\left(\vec{r}_m,t_m\right)=\delta\left(x_m\right)\delta\left(y_m\right)\delta\left(z_m+v_z t_m\right)\delta\left(t_m\right). \tag{11}$$

Let's represent it in the form of a spatially waveguide Fourier expansion:

$$Q_m\left(\vec{r}_m,t_m\right)=\iiint\frac{d\vec{K}_{m\perp}d\Omega}{(2\pi)^3}\tilde{Q}\left(\vec{K}_{m\perp},\Omega;z_m\right)\exp\left[i\left(\vec{K}_{m\perp}\vec{r}_{m\perp}-\Omega t_m\right)\right]. \tag{12}$$

Here, the integration is performed in infinite limits and the notation is used:

$$\vec{r}_{m\perp}=\left(x_m,y_m\right),\vec{K}_{m\perp}=\left(K_{mx},K_{my}\right).$$

The Fourier amplitude in (12) is defined by the expression:

$$\tilde{Q}\left(\vec{K}_{m\perp},\Omega;z_m\right)=\iiint d\vec{r}_{m\perp}dt_m Q_m\left(\vec{r}_m,t_m\right)\exp\left[-i\left(\vec{K}_{m\perp}\vec{r}_{m\perp}-\Omega t_m\right)\right]. \tag{13}$$

Substituting (11) into (13), let's obtain:

$$\tilde{Q}\left(\vec{K}_{m\perp},\Omega;z_m\right)=\int \mathrm{d}t_m\exp\left(i\Omega t_m\right)\delta\left(t_m\right)\delta\left(z_m+v_2t_m\right)=\delta\left(z_m\right). \tag{14}$$

Let's also represent the solution $G_m\left(\vec{K}_m,t_m\right)$ of the wave equation (8) in the form of the Fourier expansion:

$$G_m\left(\vec{r}_m,t_m\right)=\iiint\frac{d\vec{K}_{m\perp}dt_m}{(2\pi)^3}\tilde{G}\left(\vec{K}_{m\perp},\Omega;z_m\right)\exp\left[i\left(\vec{K}_{m\perp}\vec{r}_{m\perp}-\Omega t_m\right)\right]. \tag{15}$$

When (15) is substituted into (8), the differential operators are algebraized:

$$\frac{\partial^2}{\partial t_m^2}G_m\to -K^2\tilde{G},$$

$$-\Delta_m G_m\to K_{m\perp}^2\tilde{G}-\tilde{G}'',$$

where $K^2=\Omega^2/c_a^2$, $\tilde{G}''=\frac{\partial^2}{\partial z_m^2}\tilde{G}$.

As a result of this substitution, let's arrive at a differential equation for the Fourier amplitude $\tilde{G}\equiv\tilde{G}\left(\vec{K}_{m\perp},\Omega;z_m\right)$:

$$a_0\tilde{G}''+a_1\tilde{G}'+a_2\tilde{G}=\delta\left(z_m\right), \tag{16}$$

where $a_0=-1$, $a_1=0$, $a_2=-K_z^2\equiv-\left(K^2-K_{m\perp}^2\right)$.

Equation (16) corresponds to a special case of a stationary medium, the solution of which for $a_1\neq 0$ for a subsonic flow was considered in [5]. Using the results of [5], the solution of (16) can be written as:

$$\tilde{G}\left(\vec{K}_{m\perp},\Omega;z_m\right)=-\frac{1}{2iK_z}e^{iK_z|z_m|}\left(z_m\gtrless 0\right). \tag{17}$$

Substituting (17) into (15), let's obtain the spatiotemporal representation G_m:

$$G_m\left(\vec{r}_m,t_m\right)=\frac{i}{4\pi}\int\frac{\mathrm{d}t_m}{(2\pi)^2}e^{-i\Omega t_m}I_2, \tag{18}$$

where

$$I_2=\iint\mathrm{d}\vec{K}_{m\perp}\left(1/K_z\right)\exp\left[i\left(\vec{K}_{m\perp}\vec{r}_{m\perp}+K_z\left|z_m\right|\right)\right]=-\frac{2\pi i}{r_m}e^{iKr_m}, \tag{19}$$

according to the Weyl expansion of a spherical wave in terms of plane waves [7, 8]. Substitution (19) into (18) leads to the result:

$$G_m\left(\vec{r}_m,t_m\right)=\frac{1}{4\pi r_m}\cdot\frac{1}{2\pi}\int \mathrm{d}t_m e^{-i\Omega\left(t_m-\frac{r_m}{c_a}\right)}\equiv\frac{1}{4\pi r_m}\delta\left(t_m-\frac{r_m}{c_a}\right), \tag{20}$$

which is a well-known expression for the Green's function of an impulsive sound source in a reference frame relative to which the medium is at rest. To determine the Green's function in the frame of reference, relative to which the medium is moving, it is necessary in (20) to perform the transformation inverse to the transformation (5) $\vec{r}_m\to\vec{r}-\vec{v}t,\ t_m\to t$.

With such a transformation:

$$r_m=\sqrt{x_m^2+y_m^2+z_m^2}=\sqrt{x^2+y^2+\left(z-v_z t\right)^2}. \tag{21}$$

Taking into account (9), from (20) let's obtain:

$$G\left(\vec{r},t\right)=\frac{1}{4\pi r_m}\delta\left(g\left(t\right)\right), \tag{22}$$

where r_m is expressed in terms of $\vec{r},\ t$ according to (21).

Argument of the δ-function:

$$g\left(t\right)=t-\frac{1}{c_a}\sqrt{x^2+y^2+\left(z-v_z t\right)^2},$$

elementary transformation can be reduced to the form:

$$g\left(t\right)\equiv A\left(t\right)/B\left(t\right), \tag{23}$$

where

$$A\left(t\right)=t^2-\frac{1}{c_a^2}\left[r^2-2v_z zt+v_z^2t^2\right],\ \text{and}\ B\left(t\right)=t+\frac{1}{c_a}\sqrt{r^2-2v_z zt+v_z^2t^2}.$$

Since $A(t)$ is a quadratic polynomial, it can be represented as a product:

$$A\left(t\right)=\left(1-M_z^2\right)\left(t-t_1\right)\left(t-t_2\right), \tag{24}$$

where $t_{1,2}$ are the roots of the quadratic equation:

$$t^2+\frac{2M_z z}{c_a\left(1-M_z^2\right)}t-\frac{r^2}{c_a^2\left(1-M_z^2\right)}=0.$$

By elementary transformations, $B(t)$ can also be reduced to the form:

$$B\left(t\right)=t+\sqrt{t^2-\left(1-M_z^2\right)\left(t-t_1\right)\left(t-t_2\right)}. \tag{25}$$

Simple calculations give the following expressions for the roots $t_{1,2}$:

$$t_{1,2}=-\frac{M_z z}{c_a\left(1-M_z^2\right)}\pm\sqrt{\frac{1}{\left(1-M_z^2\right)^2}\left[\left(1-M_z^2\right)\frac{r_\perp^2}{c_a^2}+\frac{z^2}{c_a^2}\right]}, \tag{26}$$

where the upper «+» sign corresponds to the root t_1, and the lower «–» to the root t_2. Let's recall that:

$$r_\perp^2=x^2+y^2, r^2=r_\perp^2+z^2.$$

The value of the denominator $B(t)$ in (23) at $= t_{1,2}$ is:

$$B(t_{1,2}) = 2t_{1,2}, \tag{27}$$

and the numerator $A(t)$ at $t = t_{1,2}$ vanishes.

In the spherical coordinate system $r_\perp = r\sin\theta$, $z = r\cos\theta$ and formula (26) is as follows:

$$t_{1,2} = \frac{r}{c_a}\left[-\frac{M_z\cos\theta}{1-M_z^2} \pm \sqrt{\frac{1}{\left(1-M_z^2\right)^2}\left[1-M_z^2\sin^2\theta\right]}\right]. \tag{28}$$

For the case of subsonic flow $\left(M_z^2 < 1\right)$, the values of the square root in (26) are real, which leads to the values $t_1 > 0$ and $t_2 < 0$. In the case of supersonic flow $\left(M_z^2 > 1\right)$, the square root values are only valid if $1-M_z^2\sin^2\theta \geq 0$ either:

$$\sin\theta \leq \frac{1}{M_z} \equiv \sin\theta_M, \tag{29}$$

where $\theta_M = \arcsin(1/M_z)$ is the Mach angle.

In this case, both values of t_1 and t_2 are positive.

To apply the generalized Dirac δ-function in formula (22) and its consequences, it is sufficient to know its properties, and it does not matter at all from which representation this generalized function is obtained [6]. One of them for simple zeros of the function $g(t)$ is the following property:

$$\delta\left(g(t)\right) = \sum_n \frac{\delta(t-t_n)}{\left|g'(t_n)\right|}, \quad \begin{array}{l} g(t_n) = 0, \\ g'(t_n) \neq 0. \end{array}$$

To use this property of $g'(t_n)$ for $n = 1, 2$ in our case:

$$g'(t_n) = \frac{1-M_z^2}{2t_n}\left[\delta_{n1}(t_1-t_2) + \delta_{n2}(t_2-t_1)\right] + \left(1-M_z^2\right)(t_n-t_1)(t_n-t_2)\frac{\partial}{\partial t}B_{(t)}^{-1}\Big|_{t=t_n}. \tag{30}$$

Here δ_{n1}, δ_{n2} are the Kronecker symbols.

The second term in formula (30) is equal to zero, if only $\frac{\partial}{\partial t}B_{(t)}^{-1}\big|_{t=t_n}$ is a finite value:

$$\frac{\partial}{\partial t}B_{(t)}^{-1}\Bigg|_{t=t_n} = -\frac{1}{B^2(t_n)}\left[1 + \frac{2t_n - \left(1-M_z^2\right)\left[\delta_{n1}(t_1-t_2) + \delta_{\Pi 2}(t_2-t_1)\right]}{2\sqrt{t_n^2 - \left(1-M_z^2\right)(t_n-t_1)(t_n-t_2)}}\right] =$$

$$= \frac{1}{(2t_n)^3}\left[4t_n - \left(1-M_z^2\right)\left[\delta_{n1}(t_1-t_2) + \delta_{n2}(t_2-t_1)\right]\right] \neq \infty.$$

In this way,

$$g'(t_n) = \frac{1-M_z^2}{2t_n}\left[\delta_{n1}(t_1-t_2) + \delta_{n2}(t_2-t_1)\right], \ (n = 1,2). \tag{31}$$

From formulas (26), (28) it follows that:

$$(t_1-t_2) = -(t_2-t_1) = \frac{2}{c_a}\sqrt{\frac{1}{\left(1-M_z^2\right)^2}\left[z^2 + \left(1-M_z^2\right)r_\perp^2\right]} = \frac{2r}{c_a\left|1-M_z^2\right|}\sqrt{1-M_z^2\sin^2\theta}. \tag{32}$$

The value of r_m at $t = t_{1,2}$ in the denominator (22), taking into account (21) and (24), is equal to:

$$r_m\Big|_{t=t_{1,2}} = c_a\sqrt{t_n^2 - \left(1-M_z^2\right)(t_n-t_1)(t_n-t_2)} = c_a t_n, \ (n = 1,2). \tag{33}$$

and formula (22) can be rewritten as:

$$G(\vec{r},t)=\frac{1}{4\pi}\sum_{n=1,2}\frac{\delta(t-t_n)}{c_a t_n\left|g'(t_n)\right|}, \tag{34}$$

where the denominator, taking into account (32) and (33), is equal to:

$$c_a t_n\left|g'(t_n)\right|=c_a t_n\frac{\left|1-M_z^2\right|}{2t_n}\frac{2r}{c_a\left|1-M_z^2\right|}\sqrt{1-M_z^2\sin^2\theta}=r\sqrt{1-M_z^2\sin^2\theta}. \tag{35}$$

Substituting (35) into (34), let's obtain:

$$G(\vec{r},t)=\frac{1}{4\pi}\cdot\frac{1}{r\sqrt{1-M_z^2\sin^2\theta}}\left[\delta(t-t_1)+\delta(t-t_2)\right]. \tag{36}$$

In [5], formula (39) for the Green's function looks as follows:

$$G(\vec{r},\vec{r}';t,t')=\frac{1}{4\pi}\cdot\frac{1}{\hat{r}_1\sqrt{1-M^2}}\delta(t-t'-\tau).$$

When $\vec{v}=(0,0,v_z)$ there is $M^2=M_z^2$, $\vec{r}_1=\vec{r}-\vec{r}'$, $t_1=t-t'$:

$$\hat{r}_1\sqrt{1-M_z^2}=r_1\sqrt{1-M_z^2\sin^2\theta},$$

$$\tau=\frac{r_1}{c_a\left(1-M_z^2\right)}\left(-M_z\cos\theta+\sqrt{1-M_z^2\sin^2\theta}\right),$$

which coincides with t_1 in formula (28) of these calculations. The value of t_2 in a subsonic flow is negative, and therefore $\delta(t-t_2)=0$ in formula (36), which, up to the notation $t\to t_1$, $r\to r_1$, coincides with formula (39) from [5].

In supersonic flow, both values $t_1>0$ and $t_2>0$ are valid for observation points inside the Mach cone $\left(1-M_z^2\sin^2\theta>0\right)$ and, as will be seen below, correspond to observation points on the front and back of the wave front.

Observation points $\vec{r}$, at which the acoustic disturbance emitted by the source at time $t'=0$ arrives with a delay t, form a surface, the equation of which is determined by the zeros of the argument of the δ-function in (36):

$$\chi=\chi(\vec{r},t)=t-t_{1,2}(\vec{r},M_z)=0. \tag{37}$$

This surface is *a* «wave front», i.e. characteristic surface [6]. To verify this, it is necessary to calculate the first derivatives χ_t', χ_x', χ_y', χ_z' with respect to the variables t, x, y, z respectively:

$$\chi_t'=1,\ \ \chi_x'=\mp\frac{x}{c_a\left(1-M_z^2\right)R},\ \ \chi_y'=\mp\frac{y}{c_a\left(1-M_z^2\right)R},\ \ \chi_z'=+\frac{M_z}{c_a\left(1-M_z^2\right)}\mp\frac{z}{c_a\left(1-M_z^2\right)^2R},$$

where the abbreviation is used:

$$R=\sqrt{\frac{1}{\left(1-M_z^2\right)^2}\left[z^2+\left(1-M_z^2\right)r_\perp^2\right]}.$$

For the surface $\chi(\vec{r},t)=0$ to be characteristic, the calculated derivatives must satisfy identically the differential equation [6], which in our case has the form:

$$\frac{1}{c_a^2}\left(\chi_t'\right)^2-\left(\chi_x'\right)^2-\left(\chi_y'\right)^2-\left(1-M_z^2\right)\left(\chi_z'\right)^2+2\frac{M_z}{c_a}\chi_t'\chi_z'=0.$$

Substituting here the calculated derivatives, after simple calculations, it is possible to verify that this equality holds.

Using expression (26) for $t_{1,2}$ and rearranging the terms to get rid of the square root, the wavefront surface equation (37) can be reduced to the form:

$$\left[c_a t+\frac{v_z z}{c_a\left(1-M_z^2\right)}\right]^2=\frac{1}{\left(1-M_z^2\right)^2}\left[M_z^2 z^2+\left(1-M_z^2\right)r^2\right].$$

This implies:

$$\left(c_a t\right)^2=\left(r^2-2v_z zt+v_z^2t^2\right)=\left(\vec{r}-\vec{v}t\right)^2=r_m^2. \tag{38}$$

Surface (38) at time t is a sphere of radius $r_m=c_a t$, the center of which is shifted from the origin along the z axis to the point $\vec{r}_c=\vec{v}t$. The direction of the normal $\vec{n}$ at each point $\vec{r}=(x,y,z)$ at each moment of time t coincides with the direction of the radius of the sphere connecting the points $\vec{r}_c$ and $\vec{r}_m$ (**Fig. 1, 2**), therefore:

$$\vec{r}=\left(\vec{r}_c+r_m\vec{n}\right)=\left(\vec{v}+c_a\vec{n}\right)t, \tag{39}$$

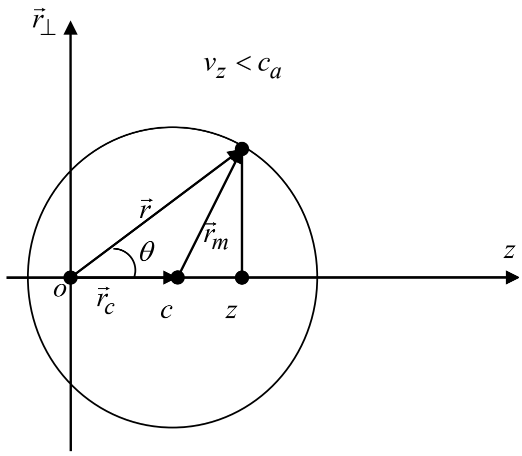

Fig. 1. Geometry of the wave front during subsonic motion of the medium

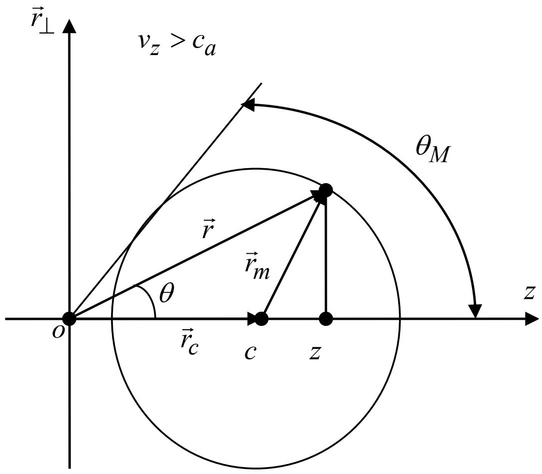

Fig. 2. Geometry of the wave front during supersonic motion of the medium

By definition [6]:

$$\frac{d\vec{r}}{dt} = (\vec{v} + c_a\vec{n}) \equiv \vec{u},$$

is the «velocity vector in the direction of the beam», and its component in the direction of the normal $\vec{n}$:

$(\vec{u}\vec{n})\vec{n} = \left[(\vec{u}\vec{n}) + c_a\right]\vec{n}$ – «vector of the wave velocity of the front of the traveling wave».

If to use formula (28) for $t_{1,2}$ in a spherical system, then for $M_z < 1$ at time t, the value $r = r(\theta)$ on the surface of the wave front is equal to:

$$r(\theta) = c_a t \cdot \frac{1 - M_z^2}{-M_z\cos\theta + \sqrt{M_z^2\cos^2\theta + 1 - M_z^2}}, \tag{40}$$

$$r(0) = c_a t + v_z t = r_{\max},$$

$$r(\pi) = c_a t - v_z t = r_{\min}.$$

For the case $M_z > 1$, a similar dependence is given by the formula:

$$r(\theta) = c_a t \cdot \frac{\left|1 - M_z^2\right|}{M_z\cos\theta \pm \sqrt{1 - M_z^2\sin^2\theta}} = r_{1,2}(\theta), \tag{41}$$

where $\left|1 - M_z^2\right| = M_z^2 - 1$, and indices 1, 2 correspond respectively to the signs «+» or «–» in the denominator.

From formula (41) it follows that:

$$r_1(0) = (v_z - c_a)t = r_{1\min},$$

$$r_2(0) = (v_z + c_a)t = r_{2\max},$$

$$r_1(\theta_M) = r_2(\theta_M) = t\sqrt{v_z^2 + c_a^2}.$$

Obviously, for angles $\theta \le \theta_M$, the value $r_2(\theta)$ corresponds to the values on the front of the wavefront, and $r_1(\theta)$ corresponds to the values on the back, which is clearly shown in **Fig. 2**.

Let's set ourselves the goal of clarifying the connection between the obtained Green's function (36) and the wave field of a point source moving in a homogeneous medium at rest. In this case, the wave operator in equation (1) for the velocity potential:

$$L = \frac{1}{c_a^2}\frac{\partial^2}{\partial t^2} - \Delta, \tag{42}$$

and the density of bulk sources:

$$Q(\vec{r}',t') = \delta(t' - t_i)\delta(\vec{r}' - \vec{r}'(t)), \tag{43}$$

where t_i is the moment of pulse emission, $\vec{r}'(t) = \vec{r}_0 + \vec{V}(t' - t_0)$ is the equation of the trajectory of the source with a velocity $\vec{V}$, $\vec{r}_0$ is the point where the source is located at the time $t = t_0$.

The Green's function for the considered case of a stationary medium is known:

$$G(\vec{r} - \vec{r}', t - t') = \frac{1}{4\pi|\vec{r} - \vec{r}'|}\delta\left(t - t' - \frac{1}{c_a}|\vec{r} - \vec{r}'|\right). \tag{44}$$

The velocity potential $\psi(\vec{r},t)$ is given by the convolution of the source density and the Green's function:

$$\psi(\vec{r},t)=\int \mathrm{d}t' \int d\,\vec{r}' G(\vec{r}-\vec{r}',t-t')Q(\vec{r}',t'). \tag{45}$$

Choosing the initial values $t_0=0$, $\vec{r}_0=0$ let's obtain

$$\psi(\vec{r},t)=\frac{1}{4\pi}\int \mathrm{d}t' \frac{\delta(t'-t_i)}{\left|\vec{r}-\vec{V}t'\right|}\delta\left[t-t'-\frac{1}{c_a}\left|\vec{r}-\vec{V}t'\right|\right]. \tag{46}$$

Usually, in the literature, the case of a moving point source, but not a pulse source, is considered [9, 10].

The presence in (46) of a multiplier in the form of a generalized $\delta(t–t_i)$-function introduces its own peculiarities into the calculation of the integral. If to introduce the replacements $\tau'=t'–t_i$ and $\tau=t–t_i$, then integration over $d\tau'=dt'$ leads to the result:

$$\psi(\vec{r},t)=\frac{1}{4\pi\left|\vec{r}-\vec{V}t_i\right|}\delta\left(\tau-\frac{1}{c_a}\left|\vec{r}-\vec{V}t_i\right|\right)\equiv\frac{1}{4\pi\left|\vec{r}_2+\vec{V}\tau\right|}\delta\big(g(\tau)\big), \tag{47}$$

where $\vec{r}_1=\vec{r}-\vec{V}t_i\equiv\vec{r}_2+\vec{V}\tau$, $\vec{r}_2=\vec{r}-\vec{V}t$, and

$$g(\tau)=\tau-\frac{1}{c_a}\left|\vec{r}_2+\vec{V}\tau\right|. \tag{48}$$

Let's transform $g(\tau)$ in the same way as it was done when obtaining formula (22):

$$g(\tau)\equiv\frac{A(\tau)}{B(\tau)},$$

where $A(\tau)=\tau^2-\frac{1}{{c_a}^2}\left|\vec{r}+\vec{V}\tau\right|^2=\left(1-\mathrm{M}_2^2\right)(\tau-\tau_1)(\tau-\tau_2)$.

Here $\mathrm{M}_2^2=V^2/c_a^2$, and $\tau_{1,2}$ are the roots of the quadratic equation:

$$\left(1-\mathrm{M}_2^2\right)\left[\tau^2-\frac{2\vec{r}_2\vec{\mathrm{M}}_2}{c_a\left(1-\mathrm{M}_2^2\right)}-\frac{{r_2}^2}{{c_a}^2\left(1-\mathrm{M}_2^2\right)}\right]=0. \tag{49}$$

By elementary transformations, the denominator $B(\tau)$ can also be transformed to the form:

$$B(\tau)=\tau+\frac{1}{c_a}\left|\vec{r}_2+\vec{V}\tau\right|\equiv\tau+\left|\tau^2-A(\tau)\right|^{1/2}.$$

For $\tau=\tau_{1,2}$ there is $B(\tau_{1,2})=2\tau_{1,2}$ and therefore:

$$g'(\tau_n)=\left(1-\mathrm{M}_2^2\right)\left[\frac{\delta_{n1}}{B(\tau_1)}(\tau_1-\tau_2)+\frac{\delta_{n2}}{B(\tau_2)}(\tau_2-\tau_1)\right].$$

Calculations similar to those performed in obtaining formulas (30)–(36) lead to the result:

$$\psi(\vec{r},t)=\frac{1}{4\pi r_2\sqrt{1-M_2^2\sin^2\theta_2}}\left[\delta(\tau-\tau_1)+\delta(\tau-\tau_2)\right], \tag{50}$$

where the roots $\tau_{1,2}$ of the quadratic equation (49) are:

$$\tau_{1,2}=\frac{\vec{r}_2\vec{M}_2}{c_a\left(1-M_2^2\right)}\pm\frac{1}{c_a}\sqrt{\frac{\left(\vec{r}_2\vec{M}_2\right)^2}{\left(1-M_2^2\right)^2}+\frac{r_2^2}{1-M_2^2}}, \tag{51}$$

$$\vec{M}_2=\vec{V}/c_a,$$

and

$$\sin^2\theta_2 = r_{2\perp}^2/r_2^2 \equiv \left(x_2^2 + y_2^2\right)/r_2^2.$$

Let's move from the reference system $\vec{r}$, t to $\vec{r}_2$, t_2, having made the Galilean transformation $\vec{r} = \vec{r}_2 + \vec{V}t_2$, $t = t_2$. The wave operator L upon transformation takes the form:

$$L_2 = \frac{1}{c_a^2}\left(\frac{\partial}{\partial t_2} - \vec{V}\vec{\nabla}_2\right)^2 - \Delta_2, \tag{52}$$

where $\vec{\nabla}_2$ and Δ_2 are the Hamilton and Laplace operators with respect to the variable $\vec{r}_2$. In order for L_2 to coincide with the operator L in (1) for a moving medium, one should choose $\vec{V} = -\vec{v}$ (the velocities in these two problems are equal in absolute value, but oppositely directed). Since when passing $\vec{r}_2$, t_2 let's obtain $\delta\left(\vec{r} - \vec{V}t\right) = \delta\left(\vec{r}_2\right)$, and $\delta(t-t_i) = \delta(t_2-t_i)$, then (taking into account the choice $\vec{r}' = 0$, $t' = 0$ specified after formula (3)) let's obtain $Q\left(\vec{r},t\right) = \delta\left(\vec{r}_2\right)\delta\left(t_2\right)$. Thus, the problem of a moving source turns into a problem of a stationary source in a moving medium.

It remains to be seen what the solution $\psi\left(\vec{r},t\right)$ defined by formula (50) will look like under such a transformation. Considering that $\vec{V} = -\vec{v}$, let's obtain $\vec{M}_2 = -\vec{M} = -\vec{v}/c_a$, and hence $\vec{r}_2\vec{M}_2 \equiv -\vec{r}_2\vec{M} = -r_2M_2\cos\theta_2$ in formula (51) for the roots $\tau_{1,2}$. Let's recall that the z-axis is directed along the vector $\vec{v}$, so $\vec{M} = \left(0, 0, M_z = M\right)$. The angle θ_2 is the angle between the z-axis and $\vec{r}_2$. Up to the notation $\vec{r} \to \vec{r}_2$, $\vec{\theta} \to \vec{\theta}_2$, the roots $\tau_{1,2}$ in formula (51) coincide with the roots $t_{1,2}$ in formula (26). Consequently, the velocity potential $\psi\left(\vec{r},t\right)$ in formula (50), with the same difference in notation, coincides with the Green's function in formula (36) for a stationary source in a moving medium.

Let's recall that formula (36) for the Green's function was obtained in a reference frame in which the origin of the Cartesian coordinate system coincides with the source $\left(\vec{r}' = 0\right)$, and the origin of time coincides with the moment of emission of the sound disturbance ($t' = 0$). In addition, the direction of the z-axis coincides with the direction of the flow velocity vector $\vec{v}$. In the frame of reference obtained as a result of the shift transformation $\vec{r}' \neq 0$, $t' \neq 0$ variables $\vec{r}$ and t in formula (36) should be replaced by $\vec{r} - \vec{r}'$ and $t-t'$, respectively, and $\cos\theta = \left(z - z'\right)/\left|\vec{r} - \vec{r}'\right|$ and $\left|\sin\theta\right| = \left|\vec{r}_\perp - \vec{r}_\perp'\right|/\left|\vec{r} - \vec{r}'\right|$ in formula (28) for $t_{1,2}$.

In this transformed form, the Green's function can be used to describe the sound field created by an arbitrary spatiotemporal distribution of sound sources. In this case, the sound field is represented as a convolution of the Green's function and the distribution density of sources (the Duhamel integral) [5]. Such a representation using the Green's function makes it possible to establish the relationship between the acoustic fields of a point high-frequency pulsed source and a monochromatic source [5], for which the acoustic pressure presented in [11] in the wave zone ($kr \gg 1$) coincides with the results presented in [2, 10].

4. Conclusions

The transition from solving the problem under consideration in a fixed IFR to solving it in an IFR moving together with the medium makes it possible to separate the process of sound drift from the process of medium oscillations under the influence of inertia and elasticity forces. This makes it possible to use the Fourier transform in the form of a spatial waveguide expansion for the solution, not only in the case of subsonic flow, which was carried out earlier, but also in the case of supersonic flow. In this latter case, such a decomposition of the solution is not feasible in the traditionally used IFR associated with a stationary source.

According to the results of the performed calculations, the difference between the obtained solutions for the subsonic and supersonic modes of motion of the medium manifests itself in the location of the moving wave front relative to the stationary sound source.

In the problem of a moving source in a stationary medium, this difference is also determined by the location of the wave front relative to the source, but moving in this case.

Let's note that the Green's function defined in this work is the exact solution of the mathematical problem formulated at the beginning of the article, and therefore does not contain any restrictions related to the calculation process. The practical use of the results of the solution is determined by how close the parameters of the medium (homogeneity and uniformity of the flow) in a particular physical situation are to the values considered in the article. For small deviations of the parameters in a certain space-time region, the results obtained can be considered as a zero approximation of the perturbation theory.

Conflict of interest

The authors declare that they have no conflict of interest in relation to this research, whether financial, personal, authorship or otherwise, that could affect the research and its results presented in this paper.

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Manuscript has no associated data.

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